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Quantum Enveloping Algebras with von Neumann Regular Cartan-like Generators and the Pierce Decomposition

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Dedicated to the memory of our colleague Leonid L. Vaksman (1951–2007)

Abstract: Quantum bialgebras derivable from $U_q(sl_2)$ which contain idempotents and von Neumann regular Cartan-like generators are introduced and investigated. Various types of antipodes (invertible and von Neumann regular) on these bialgebras are constructed, which leads to a Hopf algebra structure and a von Neumann-Hopf algebra structure, respectively. For them, explicit forms of some particular R-matrices (also, invertible and von Neumann regular) are presented, and the latter respects the Pierce decomposition.

1. Introduction

The language of Hopf algebras [1,24] is among the principal tools of studying subjects associated to noncommutative spaces [5,18] and superspaces [6,13,23] appearing as quantization of commutative ones [12,25]. An important feature of supersymmetric algebraic structures is that their underlying algebras normally contain idempotents and other zero divisors [2,10,21]. Therefore, it is reasonable to render idempotents to some quantum algebras, to study their properties and the associated Pierce decompositions [20].

In this paper we introduce a new quantum algebra which admits an embedding of $U_q\,(sl_2)$ [9,14]. After adding some extra relations we obtain two worthwhile algebras that contain idempotents and von Neumann regular Cartan-like generators. One of the algebras has the Pierce decomposition which reduces to a direct sum of two ideals and can be treated as an extended version of the algebra with von Neumann regular antipode considered in [11,17], while another one appears to be a Hopf algebra in the sense of the standard definition [1]. We distinguish some special cases for which R-matrices of

⋆ *On leave of absence from*: Theory Group, Nuclear Physics Laboratory, V. N. Karazin Kharkov National University, Svoboda Sq. 4, Kharkov 61077, Ukraine. E-mail: sduplij@gmail.com; http://webusers.physics.umn.edu/~duplij simple form are available. This way both invertible and von Neumann regular R-matrices have been produced, the latter respecting the Pierce decomposition.

2. Preliminaries

We start with recalling briefly some necessary notations and principal facts about Hopf algebras [1,4]. In our context an algebra $U^{(alg)}$ over $\mathbb{C}$ is a 4-tuple $(\mathbb{C}, A, \mu, \eta)$, where A is a vector space, $\mu : A \otimes A \to A$ is a multiplication (alternatively denoted as $\mu(a \otimes b) = a \cdot b$), $\eta : \mathbb{C} \to A$ is a unit so that $\mathbf{1} \stackrel{def}{=} \eta(1)$, $\mathbf{1} \in A$, $1 \in \mathbb{C}$. The multiplication is assumed to be associative $\mu \circ (\mu \otimes \mathsf{id}) = \mu \circ (\mathsf{id} \otimes \mu)$ and the unit is characterized by the property $\mu \circ (\eta \otimes \mathsf{id}) = \mu \circ (\mathsf{id} \otimes \eta) = \mathsf{id}$. An algebra map is a linear map $\psi : U_1^{(alg)} \to U_2^{(alg)}$ subject to $\psi \circ \mu_1 = \mu_2 \circ (\psi \otimes \psi)$ and $\psi \circ \eta_1 = \eta_2$. A coalgebra $U^{(coalg)}$ is a 4-tuple $(\mathbb{C}, C, \Delta, \epsilon)$, where C is an underlying vector space, $\Delta : C \to C \otimes C$ is a comultiplication with $\Delta(A) = \sum_i \left(A^i_{(1)} \otimes A^i_{(2)}\right)$ in the Sweedler notation, $\epsilon : C \to \mathbb{C}$ is a counit. These linear maps are subject to the following properties: coassociativity $(\Delta \otimes \mathsf{id}) \circ \Delta = (\mathsf{id} \otimes \Delta) \circ \Delta$, the counit property $(\epsilon \otimes \mathsf{id}) \circ \Delta = (\mathsf{id} \otimes \epsilon) \circ \Delta = \mathsf{id}$. A coalgebra map is a linear map $\varphi : U_1^{(coalg)} \to U_2^{(coalg)}$ such that $(\varphi \otimes \varphi) \circ \Delta_1 = \Delta_2 \circ \varphi$ and $\epsilon_1 = \epsilon_2 \circ \varphi$. A bialgebra $U^{(bialg)}$ is a 6-tuple $(\mathbb{C}, B, \mu, \eta, \Delta, \epsilon)$ which is an algebra and coalgebra simultaneously, with the compatibility conditions as follows: $\Delta \circ \mu = (\mu \otimes \mu) \circ (\mathsf{id} \otimes \tau \otimes \mathsf{id}) \circ (\Delta \otimes \Delta)$, $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, $\epsilon \circ \mu = \mu_{\mathbb{C}} \circ (\epsilon \otimes \epsilon)$, $\epsilon(\mathbf{1}) = 1$; here τ is the flip of tensor multiples, $\mu_{\mathbb{C}}$ is the multiplication in the ground field. A Hopf algebra $U^{(Hopf)}$ is a bialgebra equipped with antipode, an antimorphism of algebra subject to the relation $(S \otimes \mathsf{id}) \circ \Delta = (\mathsf{id} \otimes S) \circ \Delta = \eta \circ \epsilon$.

Let $q \in \mathbb{C}$ and $q \neq \pm 1, 0$. We start with a definition of quantum universal enveloping algebra $U_q(sl_2)$ [8]. This is a unital associative algebra $U_q^{(alg)}(sl_2)$ determined by its (Chevalley) generators k, k^{-1}, e, f, and the relations

$$k^{-1}k = \mathbf{1}, \quad kk^{-1} = \mathbf{1}, \tag{1}$$

$$ke = q^2 ek, \quad kf = q^{-2} fk, \tag{2}$$

$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}. \tag{3}$$

The standard Hopf algebra structure on $U_q^{(Hopf)}(sl_2)$ is determined by

$$\Delta_0(k) = k \otimes k, \tag{4}$$

$$\Delta_0(e) = \mathbf{1} \otimes e + e \otimes k, \; \Delta_0(f) = f \otimes \mathbf{1} + k^{-1} \otimes f, \tag{5}$$

$$\mathsf{S}_0(k) = k^{-1}, \; \mathsf{S}_0(e) = -ek^{-1}, \; \mathsf{S}_0(f) = -kf, \tag{6}$$

$$\varepsilon_0(k) = 1, \quad \varepsilon_0(e) = \varepsilon_0(f) = 0. \tag{7}$$

The algebra $U_q^{(alg)}(sl_2)$ is a domain, i.e. it has no zero divisors and, in particular, no idempotents [7,15]. A basis of the vector space $U_q(sl_2)$ is given by the monomials $k^s e^m f^n$, where $m, n \geq 0$ [14]. We denote the Cartan subalgebra of $U_q(sl_2)$ by $\mathcal{H}_0\left(\mathbf{1}, k, k^{-1}\right)$.

Our goal is to apply the Pierce decomposition to a suitably extended version of $U_q(sl_2)$. It is well known that there exists one-to-one correspondence between the central decompositions of unity on idempotents and decompositions of a module into a direct sum. Therefore we start with generalizing the Cartan subalgebra in $U_q(sl_2)$ towards the von Neumann regularity property [3,19,22].

3. From the Standard $U_q(sl_2)$ to U_{K+L}

Let us consider the generators $K, \overline{K}$ satisfying the relations

$$K\overline{K}K = K, \quad \overline{K}K\overline{K} = \overline{K}, \tag{8}$$

which are normally referred to as von Neumann regularity [19]. Under the assumption of commutativity

$$K\overline{K} = \overline{K}K \tag{9}$$

we have an idempotent $P \stackrel{def}{=} K\overline{K} = \overline{K}K$ subject to

$$PK = KP = K, \tag{10}$$

$$P^2 = P. \tag{11}$$

The commutative algebra generated by $K, \overline{K}$ is not unital (we denote it by $\mathcal{H}\left(K, \overline{K}\right)$), because unlike $U_q(sl_2)$ its relations do not anticipate unit explicitly, as in (1). Note that $\mathcal{H}\left(K, \overline{K}\right)$ was considered as a Cartan-like part of the analog of the quantum enveloping algebra with von Neumann regular antipode $U_q^v = \mathfrak{vsl}_q(2)$ introduced by Duplij and Li [11,17]. The associated unital algebra derived by an exterior attachment of unit $\mathcal{H}\left(\mathbf{1}, K, \overline{K}\right) \stackrel{def}{=} \mathcal{H}\left(K, \overline{K}\right) \oplus \mathbb{C}\mathbf{1}$ also appears in [11,17] as a part of $U_q^w = \mathfrak{wsl}_q(2)$.

Observe that $\mathcal{H}\left(\mathbf{1}, K, \overline{K}\right)$ contains one more idempotent $(\mathbf{1} - P)^2 = (\mathbf{1} - P)$. Therefore, we introduce another copy of the same algebra (we denote it by $\mathcal{H}\left(L, \overline{L}\right)$) with generators L and $\overline{L}$ subject to similar relations as for $K, \overline{K}$ above

$$L\overline{L}L - L = 0, \quad \overline{L}L\overline{L} - \overline{L} = 0. \tag{12}$$

Under the commutativity assumption

$$L\overline{L} = \overline{L}L \tag{13}$$

the idempotent $Q \stackrel{def}{=} L\overline{L} = \overline{L}L$ satisfies

$$QL = LQ = L, \tag{14}$$

$$Q^2 = Q. \tag{15}$$

If there are no additional relations between $K, \overline{K}$ and $L, \overline{L}$, the nonunital algebras $\mathcal{H}\left(K, \overline{K}\right)$ and $\mathcal{H}\left(L, \overline{L}\right)$ can form a free product only. On the other hand we merge together the unital algebras $\mathcal{H}\left(\mathbf{1}, K, \overline{K}\right)$ and $\mathcal{H}\left(\mathbf{1}, L, \overline{L}\right)$ so that their units are identified and add one more relation, the decomposition of unity

$$P + Q = \mathbf{1} \tag{16}$$

in order to produce the Pierce decomposition [20] of the resulting algebra $\mathcal{H}\left(\mathbf{1}, K, \overline{K}, L, \overline{L}\right)$, which reduces to the direct product since $QP = PQ = 0$.

It follows from (10), (14) and (16) that

$$KL = \overline{L}K = LK = K\overline{L} = \overline{K}L = L\overline{K} = 0. \tag{17}$$

The new (as compared to [11,17]) noninvertible generators $L, \overline{L}$ are introduced to justify the following

Lemma 1. *The sum $aK + bL$ is invertible, and its inverse is $a^{-1}\overline{K} + b^{-1}\overline{L}$, where $a, b \in \mathbb{R}\diagdown 0$.*

Proof. Reduces to a computation which involves (16) and (17) as

$$(aK + bL)\left(a^{-1}\overline{K} + b^{-1}\overline{L}\right) = K\overline{K} + L\overline{L} = P + Q = \mathbf{1}. \tag{18}$$

This allows us to consider a two-parameter family of morphisms for the Cartan subalgebra $\Phi_{\mathcal{H}}^{(a,b)} : \mathcal{H}_0\left(\mathbf{1}, k, k^{-1}\right) \to \mathcal{H}\left(\mathbf{1}, K, \overline{K}, L, \overline{L}\right)$ given by

$$k \to aK + bL, \qquad k^{-1} \to a^{-1}\overline{K} + b^{-1}\overline{L}. \tag{19}$$

Proposition 1. *The map $\mathbf{\Phi}_{\mathcal{H}}^{(a,b)}$ is an embedding, i.e.* $\ker \mathbf{\Phi}_{\mathcal{H}}^{(a,b)} = 0$.

Proof. Use (19) to define a homomorphism $\bar{\mathbf{\Phi}}_{\mathcal{H}}^{(a,b)}$ from the free algebra $\bar{\mathcal{H}}_0\left(\mathbf{1}, k, k^{-1}\right)$ generated by $\mathbf{1}$, k, k^{-1} into the free algebra $\bar{\mathcal{H}}\left(\mathbf{1}, K, \overline{K}, L, \overline{L}\right)$ generated by $\mathbf{1}$, K, $\overline{K}$, L, $\overline{L}$. We claim that $\bar{\mathbf{\Phi}}_{\mathcal{H}}^{(a,b)}$ is an embedding. In fact, if not, then $\bar{\mathbf{\Phi}}_{\mathcal{H}}^{(a,b)}$ annihilates some nonzero element of $\bar{\mathcal{H}}_0\left(\mathbf{1}, k, k^{-1}\right)$. This element can be treated as a "noncommutative polynomial" in three indeterminates $\mathbf{1}$, k, k^{-1}. Because the linear change of variables (19) is non-degenerate, we obtain a nontrivial polynomial in $\mathbf{1}$, K, $\overline{K}$, L, $\overline{L}$, which cannot be zero in the free algebra $\bar{\mathcal{H}}\left(\mathbf{1}, K, \overline{K}, L, \overline{L}\right)$. What remains is to observe that $\mathbf{\Phi}_{\mathcal{H}}^{(a,b)}$establishes one-to-one correspondence between the relations in $\mathcal{H}_0\left(\mathbf{1}, k, k^{-1}\right)$ and those induced on the image of $\mathbf{\Phi}_{\mathcal{H}}^{(a,b)}$, which already implies our statement for the morphism $\mathbf{\Phi}_{\mathcal{H}}^{(a,b)}$ between the quotient algebras $\mathcal{H}_0\left(\mathbf{1}, k, k^{-1}\right)$ and $\mathcal{H}\left(\mathbf{1}, K, \overline{K}, L, \overline{L}\right)$.

Now we are in a position to add two more generators E and F, along with additional relations

$$(aK + bL)\, E = q^2 E\,(aK + bL)\,, \tag{20}$$

$$\left(a^{-1}\overline{K} + b^{-1}\overline{L}\right) E = q^{-2} E\left(a^{-1}\overline{K} + b^{-1}\overline{L}\right), \tag{21}$$

$$(aK + bL)\, F = q^{-2} F\,(aK + bL)\,, \tag{22}$$

$$\left(a^{-1}\overline{K} + b^{-1}\overline{L}\right) F = q^{2} F\left(a^{-1}\overline{K} + b^{-1}\overline{L}\right), \tag{23}$$

$$EF - FE = \frac{(aK + bL) - \left(a^{-1}\overline{K} + b^{-1}\overline{L}\right)}{q - q^{-1}}, \tag{24}$$

which together with (8)-(9) and (12)-(13) determine an algebra we denote by $U_{aK+bL}^{(alg)22}$, the indices 22 stand for the numbers of generators in the left (resp., right) hand sides of the relations between the Cartan-like generators (K, L) and E, F. This algebra corresponds to $U_q^w = \mathfrak{wsl}_q\,(2)$ introduced by Duplij and Li [11,17]. To be more precise, there exists an algebra homomorphism $\mathfrak{wsl}_q\,(2) \to U_{aK+bL}^{(alg)22}$, which in the notation of [11] is given by

$$K_w \mapsto aK + bL, \qquad \overline{K}_w \mapsto a^{-1}\overline{K} + b^{-1}\overline{L}, \qquad E_w \mapsto E, \qquad F_w \mapsto F. \tag{25}$$

As one can see from Lemma 1, together with (20) – (24), the image of this homomorphism is a copy of $U_q\,(sl_2)$, cf. [11, Prop. 1].

Next we present an analog of the algebra $U_q^v = \mathfrak{vsl}_q\,(2)$ as in [11]. This is an algebra having the same generators as $U_{aK+bL}^{(alg)22}$, and being subject to the relations (together with (8) – (9) and (12) – (13)),

$$(aK + bL)\, E\left(a^{-1}\overline{K} + b^{-1}\overline{L}\right) = q^2 E, \tag{26}$$

$$\left(a^{-1}\overline{K} + b^{-1}\overline{L}\right) E\,(aK + bL) = q^{-2} E, \tag{27}$$

$$(aK + bL)\, F\left(a^{-1}\overline{K} + b^{-1}\overline{L}\right) = q^{-2} F, \tag{28}$$

$$\left(a^{-1}\overline{K} + b^{-1}\overline{L}\right) F\,(aK + bL) = q^2 F, \tag{29}$$

$$EF - FE = \frac{(aK + bL) - \left(a^{-1}\overline{K} + b^{-1}\overline{L}\right)}{q - q^{-1}}, \tag{30}$$

which we denote $U_{aK+bL}^{(alg)31}$. This algebra corresponds to the algebra $U_q^v = \mathfrak{vsl}_q\,(2)$ [11] in the sense that there exists an algebra homomorphism $\mathfrak{vsl}_q\,(2) \to U_{aK+bL}^{(alg)31}$. Again, this homomorphism, in the notation of [11], is given on the generators by (25), with the indices w being replaced by v. Another application of Lemma 1 allows one to observe that the image of this homomorphism is a copy of $U_q\,(sl_2)$, cf. [11, Prop. 1].

Introduce an extension $\mathbf{\Phi}^{(a,b)}$ of $\mathbf{\Phi}_{\mathcal{H}}^{(a,b)}$ to a morphism of $U_q\,(sl_2)$ with values in $U_{aK+bL}^{(alg)22}$ and $U_{aK+bL}^{(alg)31}$ as

$$\mathbf{\Phi}^{(a,b)} : \begin{cases} k \to aK + bL, \quad & k^{-1} \to a^{-1}\overline{K} + b^{-1}\overline{L}, \\ e \to E, & f \to F. \end{cases} \tag{31}$$

Proposition 2. *The algebras $U_{aK+bL}^{(alg)22}$ and $U_{aK+bL}^{(alg)31}$ are isomorphic to $U_{K+L}^{(alg)22} \overset{def}{=} U_{aK+bL}^{(alg)22}|_{a=1,b=1}$ and $U_{K+L}^{(alg)31} \overset{def}{=} U_{aK+bL}^{(alg)31}|_{a=1,b=1}$ respectively.*

Proof. The desired isomorphism $\Psi : U_{K+L}^{(alg)22,31} \to U_{aK+bL}^{(alg)22,31}$ is given by
$$K \to aK,\ L \to bL,\ \ \overline{K} \to a^{-1}\overline{K},\ \ \overline{L} \to b^{-1}\overline{L},\ \ E \to E,\ \ F \to F. \quad \square$$

Therefore, we will not consider the parameters a and b below.

4. Splitting the Relations

The idempotents P and Q are not central in $U_{K+L}^{(alg)22}$ and $U_{K+L}^{(alg)31}$. By allowing certain misuse of terminology, we are going to "split" the relations (20) – (24) and (26) – (30) in such a way that either P and Q become central,

$$PE = EP, \quad QE = EQ, \tag{32}$$

$$PF = FP, \quad QF = FQ, \tag{33}$$

or satisfy the "twisting" conditions

$$PE = EQ, \quad QE = EP, \tag{34}$$

$$PF = FQ, \quad QF = FP. \tag{35}$$

To be more precise, we are about to add the above relations in order to get the associated quotients of $U_{K+L}^{(alg)22}$ and $U_{K+L}^{(alg)31}$. The "splitted" 22-algebras are given by the following lists of relations:

$U_{K,L,norm}^{(alg)22}$	$U_{K,L,twist}^{(alg)22}$
$K\overline{K}K = K, \quad \overline{K}K\overline{K} = \overline{K},$ $K\overline{K} = \overline{K}K,$ $L\overline{L}L = L, \quad \overline{L}L\overline{L} = \overline{L},$ $L\overline{L} = \overline{L}L,$ $K\overline{K} + L\overline{L} = \mathbf{1},$ $KE = q^2 EK, \quad LE = q^2 EL,$ $\overline{K}E = q^{-2}E\overline{K}, \quad \overline{L}E = q^{-2}E\overline{L},$ $KF = q^{-2}FK, \quad LF = q^{-2}FL,$ $\overline{K}F = q^2 F\overline{K}, \quad \overline{L}F = q^2 F\overline{L},$ $EF - FE = \frac{(K+L) - (\overline{K}+\overline{L})}{q-q^{-1}}$	$K\overline{K}K = K, \quad \overline{K}K\overline{K} = \overline{K},$ $K\overline{K} = \overline{K}K,$ $L\overline{L}L = L, \quad \overline{L}L\overline{L} = \overline{L},$ $L\overline{L} = \overline{L}L,$ $K\overline{K} + L\overline{L} = \mathbf{1},$ $KE = q^2 EL, \quad LE = q^2 EK,$ $\overline{K}E = q^{-2}E\overline{L}, \quad \overline{L}E = q^{-2}E\overline{K},$ $KF = q^{-2}FL, \quad LF = q^{-2}FK,$ $\overline{K}F = q^2 F\overline{L}, \quad \overline{L}F = q^2 F\overline{K},$ $EF - FE = \frac{(K+L) - (\overline{K}+\overline{L})}{q-q^{-1}}$

(36)

and the "splitted" 31-algebras are defined as follows:

$U_{K,L,norm}^{(alg)31}$	$U_{K,L,twist}^{(alg)31}$
$K\overline{K}K = K, \quad \overline{K}K\overline{K} = \overline{K},$ $K\overline{K} = \overline{K}K,$ $L\overline{L}L = L, \quad \overline{L}L\overline{L} = \overline{L},$ $L\overline{L} = \overline{L}L,$ $K\overline{K} + L\overline{L} = \mathbf{1},$ $KE\overline{K} = q^2 EK\overline{K}, LE\overline{L} = q^2 EL\overline{L},$ $\overline{K}EK = q^{-2}EK\overline{K}, \overline{L}EL = q^{-2}EL\overline{L},$ $KF\overline{K} = q^{-2}FK\overline{K}, LF\overline{L} = q^{-2}FL\overline{L},$ $\overline{K}FK = q^2 FK\overline{K}, \overline{L}FL = q^2 FL\overline{L},$ $K\overline{K}(EF - FE) = \frac{K - \overline{K}}{q-q^{-1}},$ $L\overline{L}(EF - FE) = \frac{L - \overline{L}}{q-q^{-1}}$	$K\overline{K}K = K, \quad \overline{K}K\overline{K} = \overline{K},$ $K\overline{K} = \overline{K}K,$ $L\overline{L}L = L, \quad \overline{L}L\overline{L} = \overline{L},$ $L\overline{L} = \overline{L}L,$ $K\overline{K} + L\overline{L} = \mathbf{1},$ $KE\overline{L} = q^2 EL\overline{L}, LE\overline{K} = q^2 EK\overline{K},$ $\overline{K}EL = q^{-2}EL\overline{L}, \overline{L}EK = q^{-2}EK\overline{K},$ $KF\overline{L} = q^{-2}FL\overline{L}, LF\overline{K} = q^{-2}FK\overline{K},$ $\overline{K}FL = q^2 FL\overline{L}, \overline{L}FK = q^2 FK\overline{K},$ $K\overline{K}(EF - FE) = \frac{K - \overline{K}}{q-q^{-1}},$ $L\overline{L}(EF - FE) = \frac{L - \overline{L}}{q-q^{-1}}$

(37)

Note that $P = K\overline{K}$ and $Q = L\overline{L}$ are not among the generators used in (36) and (37). The relations which appear in the tables form the (equivalent) translation in terms of the "true" generators of the earlier relations for $U_{K+L}^{(alg)22}$ and $U_{K+L}^{(alg)31}$, together with the "splitting" relations (32) – (35). The procedure of deducing relations in tables from the original "non-splitted" relations in most cases reduces to right and/or left multiplication by the idempotents P and Q with subsequent use of the "annihilation rules" (17). Conversely, suppose that (36) and (37) are given. For example, let us start from the relations in the left column of (37). To see that in this case P is central, one has, using (17),

$$\begin{aligned} PE &= K\overline{K}E(P+Q) = K(\overline{K}EK)\overline{K} + K\overline{K}(EL\overline{L}) \\ &= K(q^{-2}EK\overline{K})\overline{K} + K\overline{K}(q^{-2}LE\overline{L}) = q^{-2}KE\overline{K} + 0 = EK\overline{K} = EP. \end{aligned}$$

Of course, similar ideas work also in the rest of verifications.

Proposition 3. *We have the following isomorphisms:* $U_{K,L,norm}^{(alg)22} \cong U_{K,L,norm}^{(alg)31}$, *and* $U_{K,L,twist}^{(alg)22} \cong U_{K,L,twist}^{(alg)31}$.

Proof. A straightforward computation shows that, in both cases (normal and twisted), the ideals of relations in question coincide. For instance, the right multiplication of $KE = q^2 EK$ by $\overline{K}$ in $U_{K,L,norm}^{(alg)22}$ yields $KE\overline{K} = q^2 EP$ as in $U_{K,L,norm}^{(alg)31}$. Conversely, starting from the relation $KE\overline{K} = q^2 EP$ in $U_{K,L,norm}^{(alg)31}$ we calculate $KE = K\,(PE) = K\,(EP) = \left(KE\overline{K}\right)K = \left(q^2EP\right)K = q^2 EK$ as in $U_{K,L,norm}^{(alg)22}$. Multiplying the EF-relations in $U_{K,L,norm}^{(alg)22}$, $U_{K,L,twist}^{(alg)22}$ by P and Q we obtain the EF-relations of $U_{K,L,norm}^{(alg)31}$, $U_{K,L,twist}^{(alg)31}$, and conversely, summing up the last two EF-relations of $U_{K,L,norm}^{(alg)31}$ and using (16), we obtain the EF-relations of $U_{K,L,norm}^{(alg)22}$. Similar arguments establish the second isomorphism.

Therefore, in what follows we consider the algebras $U_{K,L,norm}^{(alg)22}$, $U_{K,L,twist}^{(alg)22}$ (with the 22 superscript being discarded) only.

Now we extend the morphism $\boldsymbol{\Phi}_{\mathcal{H}}$ to that taking values in the "splitted" algebras $U_{K,L,norm}^{(alg)}$ and $U_{K,L,twist}^{(alg)}$ as follows:

$$\boldsymbol{\Phi} : \begin{cases} k \to K + L, & k^{-1} \to \overline{K} + \overline{L}, \\ \quad e \to E, & f \to F. \end{cases} \tag{38}$$

Proposition 4. *The map* $\boldsymbol{\Phi}$ *defined on the generators as above, admits an extension to a well defined morphism of algebras from* $U_q(sl_2)$ *to either* $U_{K,L,norm}^{(alg)}$ *or* $U_{K,L,twist}^{(alg)}$, *which is an embedding.*

Proof. Use an argument similar to that applied in the proof of **Proposition 1**.

Corollary 1. *Both algebras* $U_{K,L,norm}^{(alg)}$ *and* $U_{K,L,twist}^{(alg)}$ *contain* $U_q\,(sl_2)$ *as a subalgebra.*

Proof. Follows from **Proposition 4**.

Note that the Pierce decomposition of $U_{K,L,norm}^{(alg)}$ is

$$U_{K,L,norm}^{(alg)} = PU_{K,L,norm}^{(alg)}P + QU_{K,L,norm}^{(alg)}Q, \tag{39}$$

which reduces to a direct sum of the two ideals. This leads to

Proposition 5. $U_{K,L,norm}^{(alg)}$ *is a direct sum of subalgebras with each summand being isomorphic to* $U_q\,(sl_2)$.

Proof. The desired isomorphism is given by

$$K \longmapsto k \oplus 0,\quad \overline{K} \longmapsto k^{-1} \oplus 0,\quad PE \longmapsto e \oplus 0,\quad PF \longmapsto f \oplus 0, \tag{40}$$

$$L \longmapsto 0 \oplus k,\quad \overline{L} \longmapsto 0 \oplus k^{-1},\quad QE \longmapsto 0 \oplus e,\quad QF \longmapsto 0 \oplus f, \tag{41}$$

hence $P \longmapsto \mathbf{1}\oplus 0$, $Q \longmapsto 0\oplus\mathbf{1}$. This morphism splits as a direct sum of two morphisms each of the latter being, obviously, an isomorphism.

In the "twisted" case the Pierce decomposition

$$U^{(alg)}_{K,L,twist} = PU^{(alg)}_{K,L,twist}P + PU^{(alg)}_{K,L,twist}Q + QU^{(alg)}_{K,L,twist}P + QU^{(alg)}_{K,L,twist}Q, \qquad (42)$$

is nontrivial as all terms are nonzero, i.e. (42) is not a direct sum of ideals.

Let us introduce a special automorphism of algebras $U^{(alg)}_{K,L,norm}$ and $U^{(alg)}_{K,L,twist}$, which will be denoted by the same letter $\mathbf{\Upsilon}$. In either case, $\mathbf{\Upsilon}$ is given on the generators by

$$E \mapsto E,\ F \mapsto F,\ K \mapsto L,\ \overline{K} \mapsto \overline{L},\ L \mapsto K,\ \overline{L} \mapsto \overline{K},\ 1 \mapsto \mathbf{1}, \qquad (43)$$

and then extended to an endomorphism of the algebra in question. The very fact that it becomes this way a well defined linear map and then its bijectivity is established by observing that $\mathbf{\Upsilon}$ permutes the list of generators as well as the list of relations. Note that $\mathbf{\Upsilon}^2 = \mathsf{id}$.

Proposition 6. *The Poincaré-Birkhoff-Witt basis of $U^{(alg)}_{K,L,norm}$ is given by the monomials*

$$\left[\left\{PK^iE^jF^k\right\}_{i,j,k\geq 0} \cup \left\{\overline{K}^iE^jF^k\right\}_{i>0,j,k\geq 0}\right] \cup \left[\left\{QL^iE^jF^k\right\}_{i,j,k\geq 0} \cup \left\{\overline{L}^iE^jF^k\right\}_{i>0,j,k\geq 0}\right]. \qquad (44)$$

Proof. Since $U^{(alg)}_{K,L,norm}$ is a direct sum of two copies of $U_q(sl_2)$, the statement immediately follows from [14].

In the case of $U^{(alg)}_{K,L,twist}$ we have the decomposition into a direct sum of 4 vector subspaces (42). We present below a PBW basis which respects this decomposition.

Proposition 7. *The Poincaré-Birkhoff-Witt basis of $U^{(alg)}_{K,L,twist}$ is given by the monomials*

$$\left[\left\{PK^iE^jF^k\right\}_{\substack{i,j,k\geq 0\\ j+k\ even}} \cup \left\{\overline{K}^iE^jF^k\right\}_{\substack{i>0,j,k\geq 0\\ j+k\ even}}\right] \cup \left[\left\{PK^iE^jF^k\right\}_{\substack{i,j,k\geq 0\\ j+k\ odd}} \cup \left\{\overline{K}^iE^jF^k\right\}_{\substack{i>0,j,k\geq 0\\ j+k\ odd}}\right]$$
$$\cup \left[\left\{QL^iE^jF^k\right\}_{\substack{i,j,k\geq 0\\ j+k\ odd}} \cup \left\{\overline{L}^iE^jF^k\right\}_{\substack{i>0,j,k\geq 0\\ j+k\ odd}}\right] \cup \left[\left\{QL^iE^jF^k\right\}_{\substack{i,j,k\geq 0\\ j+k\ even}} \cup \left\{\overline{L}^iE^jF^k\right\}_{\substack{i>0,j,k\geq 0\\ j+k\ even}}\right]. \qquad (45)$$

Proof. It follows from (36) that the linear span of (45) is stable under multiplication by any of the generators K, $\overline{K}$, L, $\overline{L}$, E, F, which implies that this stability is also valid under multiplication by any element of $U^{(alg)}_{K,L,twist}$. Since P and Q are among the basis vectors, this linear span contains $P + Q = \mathbf{1}$, hence is just the entire algebra. To prove the linear independence of (45) it suffices to prove that every part of this vector system which is inside a specific Pierce component, is linear independent. We now stick to the special case of the Pierce component $P \cdot U_{K,L,twist}^{(alg)} \cdot P$ which is generated by the family of vectors

$$\left\{ P K^i E^j F^k \right\}_{\substack{i,j,k \geq 0 \\ j+k \ even}} \cup \left\{ \overline{K}^i E^j F^k \right\}_{\substack{i>0, j,k \geq 0 \\ j+k \ even}}, \tag{46}$$

the part of the vector system (45) inside the first bracket. Consider a (finite) linear combination

$$\sum_{\substack{i,j,k \geq 0 \\ j+k \quad even}} \alpha_{ijk} P K^i E^j F^k + \sum_{\substack{i>0, \ j,k \geq 0 \\ j+k \quad even}} \beta_{ijk} \overline{K}^i E^j F^k \tag{47}$$

which is non-trivial (not all α_{ijk} and β_{ijk} are zero). We are about to prove that (47) is non-zero. For that, we first use α_{ijk} and β_{ijk} to produce the associated non-trivial linear combination

$$\sum_{\substack{i,j,k \geq 0 \\ j+k \quad even}} \alpha_{ijk} k^i e^j f^k + \sum_{\substack{i>0 \ ,j,k \geq 0 \\ j+k \quad even}} \beta_{ijk} k^{-i} e^j f^k \tag{48}$$

in $U_q(sl_2)$. Since the monomials involved form a PBW basis in $U_q(sl_2)$ [14], the linear combination (48) is non-zero. Now apply the map $\mathbf{\Phi}$ (38) to (48) to obtain

$$\sum_{\substack{i,j,k \geq 0 \\ j+k \quad even}} \alpha_{ijk} (K+L)^i E^j F^k + \sum_{\substack{i>0, \ j,k \geq 0 \\ j+k \quad even}} \beta_{ijk} \left(\overline{K} + \overline{L} \right)^i E^j F^k. \tag{49}$$

As $\mathbf{\Phi}$ is an embedding by **Proposition 4,** we deduce that (49) is non-zero in $U_{K,L,twist}^{(alg)}$. Observe also that in the involved monomials $j+k$ is even; it follows that the projections of (49) to the Pierce components $P \cdot U_{K,L,twist}^{(alg)} \cdot Q$ and $Q \cdot U_{K,L,twist}^{(alg)} \cdot P$ are both zero. Hence (49) is the sum of its projections to $P \cdot U_{K,L,twist}^{(alg)} \cdot P$ and $Q \cdot U_{K,L,twist}^{(alg)} \cdot Q$, which are just

$$\sum_{\substack{i,j,k \geq 0 \\ j+k \quad even}} \alpha_{ijk} P K^i E^j F^k + \sum_{\substack{i>0, \ j,k \geq 0 \\ j+k \quad even}} \beta_{ijk} \overline{K}^i E^j F^k$$

and

$$\sum_{\substack{i,j,k \geq 0 \\ j+k \quad even}} \alpha_{ijk} Q L^i E^j F^k + \sum_{\substack{i>0, \ j,k \geq 0 \\ j+k \quad even}} \beta_{ijk} \overline{L}^i E^j F^k,$$

respectively. It is easy to see that these are intertwined by the automorphism $\mathbf{\Upsilon}$ (43), which implies that these projections are simultaneously zero or non-zero. Of course, the second assumption is true, because their sum (49) is non-zero. In particular,

$$\sum_{\substack{i,j,k \geq 0 \\ j+k \quad even}} \alpha_{ijk} P K^i E^j F^k + \sum_{\substack{i>0, \ j,k \geq 0 \\ j+k \quad even}} \beta_{ijk} \overline{K}^i E^j F^k$$

is non-zero, which was to be proved. The proof of linear independence of all other subsystems of (45) (in brackets), related to other Pierce components, goes in a similar way.

Let us consider the classical limit $q \to 1$ for $U_{K,L,norm}^{(alg)}$ and $U_{K,L,twist}^{(alg)}$ algebras.

Proposition 8. *The classical limit of* $U^{(alg)}_{K,L,norm}$ *is just a direct sum of two copies of classical limits for* $U_q(sl_2)$ *in the sense of [16].*

Proof. This follows from **Proposition 5**.

5. Hopf Algebra Structure and von Neumann Regular Antipode

To construct a bialgebra we need a counit on U_{K+L}, to be denoted by ε. Since P and Q are idempotents in U_{K+L}, one has $\varepsilon(P)(\varepsilon(P)-1)=0$ and $\varepsilon(Q)(\varepsilon(Q)-1)=0$, which implies that either $\varepsilon(P)=1$, $\varepsilon(Q)=0$ or $\varepsilon(P)=0$, $\varepsilon(Q)=1$. We assume the first choice. Then it follows from $L=QL$ that $\varepsilon(L)=\varepsilon(QL)=0$. Also it follows from (4) that $\varepsilon(K+L)=1$, hence $\varepsilon(K)=1$.

Elaborate the embedding $\mathbf{\Phi}$ defined in (19) and the standard relations (4), (5), (7) to transfer a coproduct onto the image of $\mathbf{\Phi}$ (31) as follows:

$$\Delta(K+L)=(K+L)\otimes(K+L), \tag{50}$$

$$\Delta\left(\overline{K}+\overline{L}\right)=\left(\overline{K}+\overline{L}\right)\otimes\left(\overline{K}+\overline{L}\right), \tag{51}$$

$$\Delta(E)=\mathbf{1}\otimes E+E\otimes(K+L), \tag{52}$$

$$\Delta(F)=F\otimes\mathbf{1}+\left(\overline{K}+\overline{L}\right)\otimes F, \tag{53}$$

$$\varepsilon(E)=\varepsilon(F)=0, \tag{54}$$

$$\varepsilon(K+L)=1, \tag{55}$$

$$\varepsilon\left(\overline{K}+\overline{L}\right)=1. \tag{56}$$

To produce a comultiplication on the above algebras $U^{(alg)}_{K,L,norm}$ and $U^{(alg)}_{K,L,twist}$ determined by (36), use (50)–(56) to define a coproduct Δ first on $\mathbf{\Phi}\left(U^{(alg)}_q(sl_2)\right)$ (via transferring from $U^{(alg)}_q(sl_2)$) and then extend it to the entire algebras $U^{(alg)}_{K,L,norm}$ and $U^{(alg)}_{K,L,twist}$ as follows:

$U^{(coalg)}_{K,L,norm}$	$U^{(coalg)}_{K,L,twist}$	
$\Delta(K)=K\otimes K,$ $\Delta(\overline{K})=\overline{K}\otimes\overline{K},$ $\Delta(L)=L\otimes L+L\otimes K+K\otimes L,$ $\Delta(\overline{L})=\overline{L}\otimes\overline{L}+\overline{L}\otimes\overline{K}+\overline{K}\otimes\overline{L},$ $\Delta(E)=\mathbf{1}\otimes E+E\otimes(K+L),$ $\Delta(F)=F\otimes\mathbf{1}+\left(\overline{K}+\overline{L}\right)\otimes F,$ $\varepsilon(E)=\varepsilon(F)=0,$ $\varepsilon(K)=1, \varepsilon(\overline{K})=1,$ $\varepsilon(L)=\varepsilon(\overline{L})=0.$	$\Delta(K)=K\otimes K+L\otimes L,$ $\Delta(\overline{K})=\overline{K}\otimes\overline{K}+\overline{L}\otimes\overline{L},$ $\Delta(L)=L\otimes K+K\otimes L,$ $\Delta(\overline{L})=\overline{L}\otimes\overline{K}+\overline{K}\otimes\overline{L}$ $\Delta(E)=\mathbf{1}\otimes E+E\otimes(K+L),$ $\Delta(F)=F\otimes\mathbf{1}+\left(\overline{K}+\overline{L}\right)\otimes F,$ $\varepsilon(E)=\varepsilon(F)=0,$ $\varepsilon(K)=1, \varepsilon(\overline{K})=1,$ $\varepsilon(L)=\varepsilon(\overline{L})=0.$	(57)

The convolution on the bialgebras $U^{(bialg)}_{K,L,norm}$ and $U^{(bialg)}_{K,L,twist}$ produced this way is defined by

$$(\mathsf{A}\star\mathsf{B})\equiv\mu(\mathsf{A}\otimes\mathsf{B})\Delta, \tag{58}$$

where A,B are linear endomorphisms of the underlying vector space.

Let us first consider the bialgebra $U_{K,L,norm}^{(bialg)}$ from the viewpoint of Hopf algebra structure.

Proposition 9. *The bialgebra $U_{K,L,norm}^{(bialg)}$ has no conventional antipode S satisfying the standard Hopf algebra axiom*

$$\mathsf{S} \star \mathsf{id} = \mathsf{id} \star \mathsf{S} = \eta \circ \varepsilon. \tag{59}$$

Proof. Since $\varepsilon(P) = 1$ and $\Delta(P) = P \otimes P$ we have from (58)

$$(\mathsf{S} \star \mathsf{id})(P) = \mathsf{S}(P)\, P = (\mathsf{id} \star \mathsf{S})(P) = P\mathsf{S}(P) = 1 \cdot \varepsilon(P) = \mathbf{1}, \tag{60}$$

which is impossible since P is not invertible.

Let us introduce an antimorphism T of $U_{K,L,norm}^{(bialg)}$ as follows:

$$\mathsf{T}(K) = \overline{K},\ \mathsf{T}\left(\overline{K}\right) = K,\ \mathsf{T}(L) = \overline{L},\ \mathsf{T}\left(\overline{L}\right) = L, \tag{61}$$

$$\mathsf{T}(E) = -E\left(\overline{K} + \overline{L}\right),\ \ \mathsf{T}(F) = -(K + L)\, F. \tag{62}$$

For $U_{K,L,norm}^{(bialg)}$ we observe that

$$(\mathsf{T} \star \mathsf{id})(K) = (\mathsf{id} \star \mathsf{T})(K) = (\mathsf{T} \star \mathsf{id})\left(\overline{K}\right) = (\mathsf{id} \star \mathsf{T})\left(\overline{K}\right) = P, \tag{63}$$

$$(\mathsf{T} \star \mathsf{id})(L) = (\mathsf{id} \star \mathsf{T})(L) = (\mathsf{T} \star \mathsf{id})\left(\overline{L}\right) = (\mathsf{id} \star \mathsf{T})\left(\overline{L}\right) = Q, \tag{64}$$

$$(\mathsf{T} \star \mathsf{id})(E) = (\mathsf{id} \star \mathsf{T})(E) = (\mathsf{T} \star \mathsf{id})(F) = (\mathsf{id} \star \mathsf{T})(F) = 0. \tag{65}$$

Proposition 10. *The antimorphism T of $U_{K,L,norm}^{(bialg)}$ is von Neumann regular*

$$\mathsf{id} \star \mathsf{T} \star \mathsf{id} = \mathsf{id}, \qquad \mathsf{T} \star \mathsf{id} \star \mathsf{T} = \mathsf{T}. \tag{66}$$

Proof. First observe that, since a convolution of linear maps is again a linear map, it suffices to verify (66) separately on the direct summands $PU_{K,L,norm}^{(bialg)}$ and $QU_{K,L,norm}^{(bialg)}$, associated to the central idempotents P and Q, respectively. We start with $PU_{K,L,norm}^{(bialg)}$, which is a sub-bialgebra. Denote by $\varphi_P : PU_{K,L,norm}^{(bialg)} \to U_q(sl_2)$ the isomorphism (40). Earlier it was introduced as an isomorphism of algebras (hence it intertwines the products, $\varphi_P \circ \mu \circ \left(\varphi_P^{-1} \otimes \varphi_P^{-1}\right) = \mu_0 = \mu_{U_q(sl_2)}$), but now it follows from (57) and $\Delta(P) = P \otimes P$ that φ_P also intertwines the comultiplication (4)-(5) of $U_q(sl_2)$ and the restriction of the comultiplication Δ of $U_{K,L,norm}^{(bialg)}$ onto $PU_{K,L,norm}^{(bialg)}$, that is, $(\varphi_P \otimes \varphi_P) \circ \Delta \circ \varphi_P^{-1} = \Delta_0$.

It follows that, given any two endomorphisms of the underlying vector space of $U_{K,L,norm}^{(bialg)}$ which leave $PU_{K,L,norm}^{(bialg)}$ invariant, then φ_P sends the convolution of them (restricted to $PU_{K,L,norm}^{(bialg)}$) to the convolution of the transferred maps on $U_q(sl_2)$.

An obvious verification shows that both id and T leave $PU_{K,L,norm}^{(bialg)}$ invariant, and then a computation shows that so do $\mathsf{id} \star \mathsf{T}$ and $\mathsf{T} \star \mathsf{id}$. Specifically, one has

$$(\mathsf{id} \star \mathsf{T})(PX) = (\mathsf{T} \star \mathsf{id})(PX) = \varepsilon_0(\varphi_P(PX))\, P$$

for any $X \in U_{K,L,norm}^{(bialg)}$. This means that φ_P establishes the equivalence of (66) on $PU_{K,L,norm}^{(bialg)}$ and the von Neumann regularity conditions for the transfer of T via φ_P on

$U_q(sl_2)$. An easy verification shows that this transfer is just S, the antipode of $U_q(sl_2)$. It is well known that S is also von Neumann regular, which finishes the proof of (66) restricted to $PU_{K,L,norm}^{(bialg)}$.

One can readily replace in the above argument φ_P by the isomorphism $\mathbf{\Phi}^{-1}$: $\mathbf{\Phi}\left(U_q(sl_2)\right) \to U_q(sl_2)$, with $\mathbf{\Phi}$ being the embedding (38). This way we obtain (66) restricted to $\mathbf{\Phi}\left(U_q(sl_2)\right)$. However, this argument is inapplicable to $QU_{K,L,norm}^{(bialg)}$, as the latter fails to be a sub-coalgebra.

Now observe that the projection of $\mathbf{\Phi}\left(U_q(sl_2)\right)$ to the direct summand $QU_{K,L,norm}^{(bialg)}$ is just $QU_{K,L,norm}^{(bialg)}$. This is because the PBW basis $\left\{k^i e^j f^k\right\}_{j,k\geq 0}$ of $U_q(sl_2)$ transferred by $\mathbf{\Phi}$ is just

$$\left\{(K+L)^i E^j F^k\right\}_{i,j,k\geq 0} \cup \left\{\left(\overline{K}+\overline{L}\right)^i E^j F^k\right\}_{i>0,j,k\geq 0}.$$

These vectors project to $QU_{K,L,norm}^{(bialg)}$ as

$$\left\{QL^i E^j F^k\right\}_{i,j,k\geq 0} \cup \left\{\overline{L}^i E^j F^k\right\}_{i>0,j,k\geq 0},$$

which form a basis in $QU_{K,L,norm}^{(bialg)}$ by **Proposition 6**. Thus, given any $X \in U_{K,L,norm}^{(bialg)}$, one can find $x \in U_q(sl_2)$ such that $QX = Q\mathbf{\Phi}(x)$. In view of this, one has

$$\begin{aligned}(\mathsf{id} \star \mathsf{T} \star \mathsf{id})(QX) &= (\mathsf{id} \star \mathsf{T} \star \mathsf{id})((\mathbf{1}-P)\mathbf{\Phi}(x)) \\ &= (\mathsf{id} \star \mathsf{T} \star \mathsf{id})(\mathbf{\Phi}(x)) - (\mathsf{id} \star \mathsf{T} \star \mathsf{id})(P\mathbf{\Phi}(x)) \\ &= \mathbf{\Phi}(x) - P\mathbf{\Phi}(x) = (\mathbf{1}-P)\mathbf{\Phi}(x) = Q\mathbf{\Phi}(x) = QX,\end{aligned}$$

due to the above observations. Certainly, a similar computation is applicable to the second part of (66), which completes its verification on $QU_{K,L,norm}^{(bialg)}$, hence on $U_{K,L,norm}^{(bialg)}$.

Definition 1. *We call the antimorphism* T *with property (66) a von Neumann regular antipode.*

Definition 2. *We call a bialgebra with a von Neumann regular antipode a von Neumann-Hopf algebra.*

Remark 1. The standard Drinfeld-Jimbo algebra $U_q(sl_2)$ (which is a domain [14]) admits no embedding of $U_{K,L,norm}^{(bialg)}$, because the latter contain zero divisors (e.g. (16)).

Let us consider a possibility to produce a Hopf algebra structure on $U_{K,L,twist}^{(bialg)}$. First we observe that the argument of the proof of **Proposition 9** does not work in this case. Indeed, an application of (59) to P yields, instead of (60), the following relation:

$$\mathsf{S}(P)P + \mathsf{S}(Q)Q = \mathbf{1}, \tag{67}$$

which does not contradict to noninvertibility of P and Q as in the context of (60). Introduce an antimorphism S of $U_{K,L,twist}^{(bialg)}$ by the same formulas as (61)–(62),

$$\mathsf{S}(K) = \overline{K}, \mathsf{S}\left(\overline{K}\right) = K, \mathsf{S}(L) = \overline{L}, \mathsf{S}\left(\overline{L}\right) = L, \tag{68}$$

$$\mathsf{S}(E) = -E\left(\overline{K}+\overline{L}\right), \quad \mathsf{S}(F) = -(K+L)F. \tag{69}$$

We have for $U_{K,L,twist}^{(bialg)}$,

$$(\mathsf{id}\star\mathsf{S})(K)=(\mathsf{S}\star\mathsf{id})(K)=(\mathsf{S}\star\mathsf{id})\left(\overline{K}\right)=(\mathsf{id}\star\mathsf{S})\left(\overline{K}\right)=\mathbf{1}, \tag{70}$$

$$(\mathsf{id}\star\mathsf{S})(L)=(\mathsf{S}\star\mathsf{id})(L)=(\mathsf{S}\star\mathsf{id})\left(\overline{L}\right)=(\mathsf{id}\star\mathsf{S})\left(\overline{L}\right)=0, \tag{71}$$

$$(\mathsf{id}\star\mathsf{S})(E)=(\mathsf{S}\star\mathsf{id})(E)=(\mathsf{S}\star\mathsf{id})(F)=(\mathsf{id}\star\mathsf{S})(F)=0. \tag{72}$$

The proof of the following statement is basically due to [14, p.35].

Proposition 11. *The relations* $(\mathsf{id}\star\mathsf{S})(X)=(\mathsf{S}\star\mathsf{id})(X)=\varepsilon(X)\cdot\mathbf{1}$ *are valid for any* $X\in U_{K,L,twist}^{(bialg)}$.

Proof. Note that $X\mapsto\varepsilon(X)\mathbf{1}$ is a morphism of algebras. Hence, in view of an obvious induction argument, it suffices to verify that $(\mathsf{id}\star\mathsf{S})(XY)=(\mathsf{id}\star\mathsf{S})(X)\cdot(\mathsf{id}\star\mathsf{S})(Y)$ and $(\mathsf{S}\star\mathsf{id})(XY)=(\mathsf{S}\star\mathsf{id})(X)\cdot(\mathsf{S}\star\mathsf{id})(Y)$, with X being one of the generators $K,\overline{K},L,\overline{L},E,F$ and Y arbitrary. We use the Sweedler notation $\Delta(X)=\sum_i X_i'\otimes X_i''$ [24] to get

$$(\mathsf{S}\star\mathsf{id})(XY)=\sum_{ij}\mathsf{S}\left(Y_j'\right)\mathsf{S}\left(X_i'\right)X_i''Y_j''.$$

It follows from (70)–(72) that $\sum_i\mathsf{S}\left(X_i'\right)X_i''$ is a scalar multiple of $\mathbf{1}$, hence is central in $U_{K,L,twist}^{(bialg)}$, and we obtain

$$(\mathsf{S}\star\mathsf{id})(XY)=\sum_{ij}\mathsf{S}\left(X_i'\right)X_i''\mathsf{S}\left(Y_j'\right)Y_j''$$

$$=\left(\sum_i\mathsf{S}\left(X_i'\right)X_i''\right)\left(\sum_j\mathsf{S}\left(Y_j'\right)Y_j''\right)=(\mathsf{S}\star\mathsf{id})(X)\cdot(\mathsf{S}\star\mathsf{id})(Y).$$

Of course, a similar argument goes also for $(\mathsf{id}\star\mathsf{S})$.

Thus, we have the following

Theorem 1. 1) $U_{K,L}^{(Hopf)}\overset{def}{=}\left(U_{K,L,twist}^{(bialg)},\mathsf{S}\right)$ *is a Hopf algebra;*

2) $U_{K,L}^{(vN-Hopf)}\overset{def}{=}\left(U_{K,L,norm}^{(bialg)},\mathsf{T}\right)$ *is a von Neumann-Hopf algebra.*

6. Structure of *R*-matrix and the Pierce Decomposition

Let us consider a version of the universal R-matrix for $U_{K,L}^{(vN-Hopf)}$ and $U_{K,L}^{(Hopf)}$. In order to avoid considerations related to formal series (the general context of R-matrices), we turn to quasi-cocommutative bialgebras [16]. Such bialgebras generate R-matrices of some simpler shape admitting (under some additional assumptions) an explicit formula to be described below.

Definition 3. *A bialgebra* $U^{(bialg)}=(\mathbb{C},B,\mu,\eta,\Delta,\varepsilon)$ *is called quasi-cocommutative, if there exists an invertible element* $R\in U^{(bialg)}\otimes U^{(bialg)}$*, called a universal* R*-matrix, such that*

$$\Delta^{cop}(b)=R\Delta(b)R^{-1},\quad\forall b\in U^{(bialg)}, \tag{73}$$

where Δ^{cop} *is the opposite comultiplication in* $U^{(bialg)}$.

The R-matrix of a braided bialgebra $U^{(bialg)}$ is subject to

$$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}, \quad (\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}, \tag{74}$$

where for $R = \sum_i s_i \otimes t_i$ one has $R_{12} = \sum_i s_i \otimes t_i \otimes \mathbf{1}$, etc. [9]. From now on we assume that $q^n = 1$, which is a distinct case in the above context.

Consider the two-sided ideal I_{sl_2} in $U_q^{(alg)}(sl_2)$ generated by $\{k^n - \mathbf{1}, e^n, f^n\}$, together with the associated quotient algebra $\widehat{U}_q^{(alg)}(sl_2) = U_q^{(alg)}(sl_2) \diagup I_{sl_2}$.

Theorem 2 ([16, p.230]). *The universal R-matrix of* $\widehat{U}_q^{(alg)}(sl_2)$ *is*

$$\widehat{R} = \sum_{0 \le i,j,m \le n-1} A_m^{ij}(q) \cdot e^m k^i \otimes f^m k^j, \tag{75}$$

$$A_m^{ij}(q) = \frac{1}{n} \frac{(q - q^{-1})^m}{[m]!} q^{\frac{m(m-1)}{2} + 2m(i-j) - 2ij}, \tag{76}$$

where $[m]! = [1][2]\dots[m]$, $[m] = \left(q^m - q^{-m}\right) \diagup \left(q - q^{-1}\right)$.

Now we use (38) to obtain an analog of this theorem for $U_{K,L}^{(Hopf)}$. In a similar way we consider the quotient algebra $\widehat{U}_{K+L}^{(Hopf)} = U_{K,L}^{(Hopf)} \diagup I_{K+L}^{(Hopf)}$, where the two-sided ideal $I_{K+L}^{(Hopf)}$ is generated by $\{K^n + L^n - \mathbf{1}, E^n, F^n\}$.

Theorem 3. *The universal R-matrix of* $\widehat{U}_{K,L}^{(Hopf)}$ *is given by*

$$\widehat{R}_{K+L}^{(Hopf)} = \sum_{0 \le i,j,m \le n-1} A_m^{ij}(q) \cdot E^m \left(K^i + L^i\right) \otimes F^m \left(K^j + L^j\right). \tag{77}$$

Proof. In view of the morphism $\widehat{\mathbf{\Phi}} : \widehat{U}_q^{(alg)}(sl_2) \to \widehat{U}_{K+L}^{(Hopf)}$ induced by (38) and **Theorem 2**, it suffices (due to invertibility of R) to verify the relation $\Delta^{cop}(b)\,\widehat{R}_{K+L}^{(Hopf)} = \widehat{R}_{K+L}^{(Hopf)} \Delta(b)$ for $b = K, \overline{K}$, because Δ and Δ^{cop} are morphisms of algebras. This claim reduces to the verification of

$$\begin{aligned}&(K \otimes K + L \otimes L)\left(E^m \left(K^i + L^i\right) \otimes F^m \left(K^j + L^j\right)\right)\\ &= \left(E^m \left(K^i + L^i\right) \otimes F^m \left(K^j + L^j\right)\right)(K \otimes K + L \otimes L),\end{aligned} \tag{78}$$

and

$$\begin{aligned}&\left(\overline{K} \otimes \overline{K} + \overline{L} \otimes \overline{L}\right)\left(E^m \left(\overline{K}^i + \overline{L}^i\right) \otimes F^m \left(\overline{K}^j + \overline{L}^j\right)\right)\\ &= \left(E^m \left(\overline{K}^i + \overline{L}^i\right) \otimes F^m \left(\overline{K}^j + \overline{L}^j\right)\right)\left(\overline{K} \otimes \overline{K} + \overline{L} \otimes \overline{L}\right),\end{aligned} \tag{79}$$

using (36). The relations (74) are transferred by $\widehat{\mathbf{\Phi}}$ into our picture, because $\widehat{R}_{K+L}^{(Hopf)}$ is inside of the tensor square of the image of $\widehat{\mathbf{\Phi}}$.

Turn to writing down an explicit form for the universal R-matrix in the case of $U_{K,L}^{(vN-Hopf)}$. Again we consider the quotient algebra $\widehat{U}_{K+L}^{(vN-Hopf)} = U_{K,L}^{(vN-Hopf)} \diagup I_{K+L}^{(vN-Hopf)}$, where the two-sided ideal $I_{K,L}^{(vN-Hopf)}$ is generated by $\{K^n + L^n - \mathbf{1}, E^n, F^n\}$.

Theorem 4. *The universal R-matrix of* $\widehat{U}_{K+L}^{(vN-Hopf)}$ *is given by*

$$\widehat{R}_{K+L}^{(vN-Hopf)} = \sum_{0\leq i,j,m\leq n-1} A_m^{ij}(q)\cdot E^m\left(K^i + L^i\right)\otimes F^m\left(K^j + L^j\right). \tag{80}$$

Proof. Is the same as that of **Theorem 3**.

Remark 2. In view of **Theorem 2** the R-matrices we have introduced satisfy the Yang-Baxter equation by our construction.

Note that $\widehat{R}_{K+L}^{(vN-Hopf)}$ is not submitted to the direct sum decomposition (39). Now we present another notion of R-matrix which respects (39), but differs from that described in **Definition 3** in the sense of being noninvertible.

Definition 4. *A bialgebra* $\widetilde{U}^{(bialg)} = (\mathbb{C}, B, \mu, \eta, \Delta, \varepsilon)$ *is called near-quasi-cocommutative, if there exists an element* $\widetilde{R} \in \widetilde{U}^{(bialg)} \otimes \widetilde{U}^{(bialg)}$*, called a universal near-R-matrix, such that*

$$\Delta^{cop}(b)\,\widetilde{R} = \widetilde{R}\Delta(b), \quad \forall b \in \widetilde{U}^{(bialg)}, \tag{81}$$

where Δ^{cop} *is the opposite comultiplication in* $\widetilde{U}^{(bialg)}$ *and an element* $\widetilde{R}^{\dagger} \in \widetilde{U}^{(bialg)} \otimes \widetilde{U}^{(bialg)}$ *is such that*

$$\widetilde{R}\widetilde{R}^{\dagger}\widetilde{R} = \widetilde{R}, \quad \widetilde{R}^{\dagger}\widetilde{R}\widetilde{R}^{\dagger} = \widetilde{R}^{\dagger}, \tag{82}$$

and $\widetilde{R}^{\dagger}$ *can be named the Moore-Penrose inverse for a near-R-matrix [19,22].*

A near-quasi-cocommutative bialgebra $\widetilde{U}^{(bialg)}$ is braided, if its near-R-matrix satisfies (74).

Consider the quotient algebra $\widehat{U}_{K,L}^{(vN-Hopf)} = U_{K,L}^{(vN-Hopf)} \diagup I_{K,L}^{(vN-Hopf)}$, where the two-sided ideal $I_{K,L}^{(vN-Hopf)}$ is generated by $\{K^n - P, L^n - Q, E^n, F^n\}$.

Theorem 5. *The universal R-matrix of* $\widehat{U}_{K,L}^{(vN-Hopf)}$ *is given by the sum*

$$\widehat{R}_{K,L}^{(vN-Hopf)} = \widehat{R}_{PP}^{(vN-Hopf)} + \widehat{R}_{QQ}^{(vN-Hopf)}, \tag{83}$$

where

$$\widehat{R}_{PP}^{(vN-Hopf)} = \sum_{0\leq i,j,m\leq n-1} A_m^{ij}(q)\cdot E^m K^i \otimes F^m K^j, \tag{84}$$

$$\widehat{R}_{QQ}^{(vN-Hopf)} = \sum_{0\leq i,j,m\leq n-1} A_m^{ij}(q)\cdot E^m L^i \otimes F^m L^j. \tag{85}$$

Remark 3. The universal near-R-matrix $\widehat{R}_{K,L}^{(vN-Hopf)}$ can be presented in the form

$$\widehat{R}_{K,L}^{(vN-Hopf)} = (P\otimes P)\,\widehat{R}_{PP}^{(vN-Hopf)} + (Q\otimes Q)\,\widehat{R}_{QQ}^{(vN-Hopf)}. \tag{86}$$

Proof. Recall that $U_{K,L}^{(vN-Hopf)}$ admits the direct sum decomposition (39) with each summand being isomorphic to $U_q(sl_2)$. After dividing out by the ideal $I_{K,L}^{(vN-Hopf)}$ we get

$$\widehat{U}_{K,L}^{(vN-Hopf)} = PU_{K,L}^{(vN-Hopf)}P \diagup \left\{ I_{K,L}^{(vN-Hopf)} \cap PU_{K,L}^{(vN-Hopf)}P \right\} + QU_{K,L}^{(vN-Hopf)}Q \diagup \left\{ I_{K,L}^{(vN-Hopf)} \cap QU_{K,L}^{(vN-Hopf)}Q \right\}. \tag{87}$$

Each of the summands of the right hand side of (87) is clearly isomorphic to $\widehat{U}_q^{(alg)}(sl_2)$, and the isomorphisms in question take $\mathbf{1} \in \widehat{U}_q^{(alg)}(sl_2)$ to P and Q respectively. Now it follows from **Theorem 2**, that each of the terms of (86) satisfies the conditions of **Definition 3** and (74), hence so does their sum $\widehat{R}_{K,L}^{(vN-Hopf)}$. Also it follows from **Theorem 2**, that there exist $\widehat{R}_{PP}^{(vN-Hopf)\dagger}$, $\widehat{R}_{QQ}^{(vN-Hopf)\dagger} \in \widehat{U}_{K,L}^{(vN-Hopf)} \otimes \widehat{U}_{K,L}^{(vN-Hopf)}$ such that

$$\widehat{R}_{PP}^{(vN-Hopf)}\widehat{R}_{PP}^{(vN-Hopf)\dagger} = \widehat{R}_{PP}^{(vN-Hopf)\dagger}\widehat{R}_{PP}^{(vN-Hopf)} = P \otimes P, \tag{88}$$

$$\widehat{R}_{QQ}^{(vN-Hopf)}\widehat{R}_{QQ}^{(vN-Hopf)\dagger} = \widehat{R}_{QQ}^{(vN-Hopf)\dagger}\widehat{R}_{QQ}^{(vN-Hopf)} = Q \otimes Q, \tag{89}$$

hence the von Neumann regularity (82) is valid for

$$\widehat{R}^{(vN-Hopf)} = \widehat{R}_{PP}^{(vN-Hopf)} + \widehat{R}_{QQ}^{(vN-Hopf)}, \tag{90}$$

because $\widehat{R}_{PP}^{(vN-Hopf)}$, $\widehat{R}_{PP}^{(vN-Hopf)\dagger}$ and $\widehat{R}_{QQ}^{(vN-Hopf)}$, $\widehat{R}_{QQ}^{(vN-Hopf)\dagger}$ are mutually orthogonal.

7. Conclusion

Thus, we have introduced a couple of new bialgebras derived from $U_q(sl_2)$ which contain idempotents (hence some zero divisors). In some special cases explicit formulas for R-matrices are presented. We define near-R-matrices which satisfy the von Neumann regularity condition.

In a similar way one can consider an analog of $U_q(sl_n)$ furnished by a suitable and more cumbersome family of idempotents. Also, it would be worthwhile to investigate supersymmetric versions of the presented structures.

Hopefully, this approach will be able to facilitate a further research of bialgebras splitting into direct sums, which is a new way of generalizing the standard Drinfeld-Jimbo algebras.

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[1] Memorial Page: http://webusers.physics.umn.edu/~duplij/vaksman

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